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On the two-dimensional viscous counterpart of the one-dimensional sonic throat

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An analysis of the full, compressible, non-adiabatic boundary-layer equations is presented to describe the so-called 'throat' formed when a two-dimensional viscous layer, interacting with a supersonic inviscid outer stream, is accelerated or decelerated through sonic velocity defined in some mean sense. The basic analysis differs from previous momentum integral theories in that the dynamics of the viscous layer is described by the exact local expressions for the streamwise gradients of the flow variables that obtain from the boundary-layer conservation equations, rather than on streamwise derivatives of integral properties of these equations. The theory is then used to develop an extensive analogy with the classical analysis of the throat in the inviscid quasi one-dimensional streamtube. The theory shows that a single integral constraint exists at the throat, which relates the velocity and temperature profiles in the viscous layer to the motion of the inviscid outer flow. One consequence of this constraint is that, for a one-parameter family of profile shapes, the solution can be started at the throat station by specifying only a single variable, the free stream Reynolds number based on the physical thickness of the viscous layer at the throat station. For the hypersonic near wake, this simplification permits one to obtain an approximate solution for the downstream flow without first solving the detailed motion in the base recirculation region. The paper ends with a discussion of the numerical results for the Stewartson family of wake-like profiles.

1. Introduction

One of the simplest, yet most fundamental and useful of the mathematical treatments in the historical development of modern gas dynamics is the analysis of the steady, quasi one-dimensional, inviscid streamtube. By far the most intriguing and important aspect of this problem is the change in physical behaviour that occurs at the station where the flow is accelerated or decelerated through sonic velocity. A closely related problem is the corresponding two-dimensional phenomenon that occurs when a viscous layer, interacting with a

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supersonic outer stream, is accelerated or decelerated through sonic velocity described in some gross sense. This problem has been the subject of much recent interest because of its relation to supersonic wake flows. Due to the complexity of the governing equations for a shear layer, most previous investigators have appealed to integral techniques to study the two-dimensional viscous phenomenon. In the present paper a more general theory is developed to treat this phenomenon, which has much of the same rigor and simplicity as the classical analysis of the inviscid one-dimensional throat. This theory, since it does not employ averaging approximations but uses instead the local expressions for the streamwise gradients of the flow variables as they obtain from the boundary-layer conservation equations, is able to provide an understanding of the detailed behaviour and answer a number of conceptual questions that are inaccessible to the integral techniques. The limitations of the analysis are basically those inherent in the boundary-layer approximation: the neglect of the normal pressure field and the arbitrariness in the definition of the boundary-layer outer edge.

The integral techniques, first introduced by Crocco & Lees (1952) for adiabatic flow and since elaborated on and extended by a number of investigators (Glick 1962; Lees & Reeves 1964; Reeves & Lees 1965; Webb *et al.* 1965; Golik, Webb & Lees 1967; Ai 1967), have shown important qualitative similarities between the over-all behaviour of the one-dimensional and two-dimensional phenomena. In these theories the viscous region, by virtue of the integral averaging process, is represented as an equivalent one-dimensional flow with dissipation. The interaction with the inviscid outer flow enters through the description of the streamwise pressure gradient, which is not prescribed as in ordinary boundary-layer theory but determined through the growth in displacement thickness of the viscous region.

The above investigations have all shown that for profiles typical of the laminar near wake the downstream integration of the coupled system of ordinary differential equations for the integral average properties encounters an unstable singularity of saddle-point type. The solutions exhibit a divergent downstream behaviour that can be suppressed through very fine adjustments of a single free parameter in the initial data. This singularity, often referred to as the Crocco–Lees 'critical point', is similar in several respects to the saddle point singularity at the $M = 1$ or minimum area station in the inviscid, quasi one-dimensional streamtube. While a rigorous analogy is not possible, the results of the integral method suggest that at the 'critical point' the Mach number of the viscous layer is unity in some average sense. At this station a complicated mathematical constraint is imposed on the inviscid outer flow which limits the possible outer flow solutions to a discrete number (Ai 1967) but does not necessarily uniquely determine the flow. Intuitively, one suspects that this constraint is related in some manner to the corresponding constraint, $dA/dx = 0$, at the throat of the quasi one-dimensional streamtube. It is also thought that at the 'critical point' the physical behaviour changes from subsonic to supersonic in the mean, or *vice versa*, with respect to both the pressure-area rule and the upstream propagation of disturbances. The latter behaviour is suggested by the fact that when the forward integration of the integral relations is initiated downstream of the 'critical point' the divergent behaviour mentioned earlier is no longer present. A more precise treatment of each of these points can be obtained from the detailed analogy with the quasi one-dimensional streamtube developed in the present study.

The results obtained from the integral moment methods have raised several important conceptual questions. One question involves the relationship of the throat constraint to the problem of flow uniqueness. The results of the integral methods indicate that for a one-parameter family of profile shapes the condition that the solution curves pass smoothly through the downstream singularity is sufficient to determine the one free parameter. An obvious corollary to this is that for a many-parameter description this same condition cannot by itself determine the profiles at the singularity. These results are also borne out by the present theory, which shows that the throat constraint takes the form of a single integral condition relating the physical thickness of the viscous layer and the various parameters used in the description of the velocity and enthalpy profiles.

A second question arising from the results of the integral methods concerns the frequent absence of any solution for sufficiently low Reyonolds number. Ai (1967) has carefully analyzed the 'critical point singularity' of the multi-moment integral theory in Poincare phase space. He found that, for all cases studied, there is a minimum critical Reynolds number below which no solutions exist. It is not clear from Ai's analysis whether this behaviour is physically meaningful, or due to approximations inherent in the integral method.

A third basic question is raised by the dependence of the results on the somewhat arbitrary definition of the boundary-layer outer edge. Weinbaum (1967) has suggested that the 'critical point' might even be avoided entirely by suitably redefining the boundary-layer outer edge. While the present study shows that this suggestion is false for all situations encountered in practice, the definition of δ remains one of the intrinsic weaknesses of the theory. This state of affairs is somewhat clarified in § 5, where the sensitivity of the throat profiles to the choice of δ is explored numerically.

Perhaps the most important potential application of the throat problem for supersonic wake flows is the possibility of formulating a much simpler initial value problem at the wake throat *per se*. In the event that one is able to identify possible wake solutions right at the throat station, and that disturbances downstream of the throat do not influence conditions upstream, this simplification would be of much practical value. Heretofore the major difficulties in generating a suitable set of initial profiles for a far wake calculation were (i) the need for first determining the detailed motion of the complicated flow in the base recirculation region and (ii) the difficulty of integrating the equations forward through the throat singularity. Obviously both these problems could be avoided if, to some approximation, a unique initial value problem could be defined at the throat location. That one could start the solution at the throat with sufficient information given is obvious too. The non-trivial problem is to determine the minimum amount of information that need be given to yield solutions of a prescribed accuracy, to pose the problem in such a way that this given information can be obtained without recourse to a detailed solution of the base flow region, and to determine which pressure gradient at the throat station is compatible with the profile shapes that have been determined by the given information and the throat constraint, so as to be able to start the solution. For the Laval nozzle the latter problem is simply resolved by relating the pressure gradient at the throat to the curvature of the nozzle wall at the throat station. The present problem is more difficult, since the curvature of the streamlines at the edge of the layer depends in a complex manner on both the detailed mechanisms taking place within the shear layer and its interaction with the inviscid outer stream. The solution to this problem is presented in §3 (iii).

Finite difference solutions of the higher-order boundary-layer equations with free interaction, starting at the wake rear-stagnation point, have also been performed (Baum & Denison 1966). Although the governing equations in this investigation are partial differential equations and not ordinary differential equations as in the integral methods, the same saddle-point type behaviour is observed. For the equations to be integrated smoothly through the downstream throat one must solve by trial and error an eigenvalue problem for the initial outer edge Mach number in the inviscid flow. The interesting feature of this work is that the finite difference solution did not indicate a different saddle point for each streamtube at its own throat but rather a single downstream singularity as in the integral method. No explanation is offered as to why the throat is still apparently a one-dimensional rather than a two-dimensional phenomenon. The theory in §3 clarifies this point.

A final topic of interest, alluded to earlier, is the possible change in the upstream influence properties of the viscous-layer equations with interaction at the throat station. The linear analysis of Lighthill (1950, 1953*a*, *b*) for the behaviour of small disturbances in a viscous layer has shown that the free interaction bounpary condition introduces certain elliptic features into the problem. More recently, Garvine (1968) has developed a different approach based on an Oseen linearization to treat interaction problems of this type. A discussion of the implications of both these analyses for the throat phenomenon is presented in the last part of §3.

Section 2 provides a capsule summary of the principal results for the inviscid quasi one-dimensional streamtube. Section 3 presents the general theory for the two-dimensional viscous layer and develops the detailed analogy with each of the key results outlined in §2. Section 4 discusses the selection of the throat profiles and the necessary conditions for a laminar supersonic wake flow to be formulated as an initial value problem at the wake throat. Sample numerical calculations are then shown in §5 for a typical family of wake-like profiles. Section 6 enumerates the more important general results.

2. The inviscid quasi one-dimensional streamtube

The key results that obtain at the throat of the inviscid quasi one-dimensional streamtube are briefly summarized and interpreted below in order to elucidate the parallel development and detailed analogy that will be presented for the two-dimensional viscous throat in succeeding sections. The inviscid equation relating the differential changes in pressure and area for the quasi one-dimensional streamtube, suitably non-dimensionalized, appears as follows (Liepmann & Roshko 1957, p. 52). Here M is the Mach number, p the pressure, A the streamtube cross-sectional area, x a co-ordinate measured along the axis of the streamtube and γ the gas specific heat ratio:

$$\frac{M^2-1}{\gamma p M^2}\frac{dp}{dx} = -\frac{1}{A}\frac{dA}{dx}. \tag{2.1}$$

The analogy follows directly from the detailed comparison of the behaviour of (2.1) and the behaviour of its two-dimensional counterpart (3.7). For each of the principal results described below for the one-dimensional throat one anticipates a two-dimensional equivalent.

(*a*) First, one notes the change in area rule that occurs at $M = 1$. When $dp/dx > 0$, dA/dx is >0 or <0 for $M < 1$ or >1 in that order. The reverse is true for $dp/dx < 0$.

(*b*) If dp/dx is to be bounded at the station where

$$M = 1, \tag{2.2}$$

then

$$\frac{dA}{dx} = 0, \tag{2.3}$$

and in view of (*a*) this station must also be a minimum area station; that is, $d^2A/dx^2 \geqslant 0$. Only when $M = 1$ can dp/dx be non-zero if $dA/dx = 0$.

(*c*) The value of the pressure gradient at the throat station is not arbitrary but is related to the throat curvature d^2A/dx^2. If one divides (2.1) through by $(M^2-1)/\gamma p M^2$ the resulting expression for dp/dx is of 0/0 form at $M = 1$. This indeterminate form can be evaluated through the application of L'Hopital's rule. The x-derivatives of the flow variables that arise in the limiting process can be related to dp/dx. This procedure yields a quadratic equation for dp/dx whose solutions are

$$\frac{dp}{dx} = \pm\gamma p^* \left[\frac{\dfrac{d^2A}{dx^2}}{(\gamma+1)A^*}\right]^{\frac{1}{2}} \quad \text{at} \quad M = 1. \tag{2.4}$$

Here the star superscript denotes conditions at $M = 1$. If dp/dx is to be real, $d^2A/dx^2 \geqslant 0$. The positive and negative roots lead to subsonic and supersonic flows downstream of the throat in that order. Only at $M = 1$ can dual solutions exist and a downstream branching be initiated.

(*d*) Another important feature of the sonic throat is the change in upstream influence characteristics of the flow at this station. Once the flow has been accelerated to sonic velocity at the throat or minimum area station, a decrease in downstream pressure cannot influence conditions at, or upstream of, the throat location. In this sense, the flow at the throat is uniquely determined. The uniqueness of the throat solution is an important simplification in nozzle design, for it provides a simple yet realistic set of initial conditions for determining the actual two-dimensional motion in the diverging portion of a supersonic nozzle. As we shall see in § 5, a related simplification obtains for a shear layer if the u and h profiles are limited to a one-parameter family.

3. Basic relations for the two-dimensional viscous layer

As suggested above, it is possible to derive directly from the full compressible, non-adiabatic Prandtl boundary-layer equations, without further approximation, all the equivalent results for a two-dimensional viscous layer that were summarized in §2 for the quasi one-dimensional streamtube. A parallel development with §2 will now be pursued.

The first objective is to derive the pressure-area relation for the inner viscous layer that corresponds to the inviscid streamtube pressure-area relation given by (2.1). First, the governing equations for a compressible boundary layer are written. In non-dimensional form they appear as

$$(\rho u)_x = -(\rho v)_y, \tag{3.1}$$

$$\rho u u_x + \frac{1}{\gamma M_\infty^2}\frac{dp_e}{dx} = -\rho v u_y + \frac{1}{R_{\infty L}}(\mu u_y)_y, \tag{3.2}$$

$$\rho u h_x - \left(\frac{\gamma-1}{\gamma}\right) u \frac{dp_e}{dx} = -\rho v h_y + \frac{1}{R_{\infty L} P}(\mu h_y)_y + \frac{\gamma-1}{R_{\infty L}} M_\infty^2 \mu u_y^2, \tag{3.3}$$

$$p_e = \rho h, \tag{3.4}$$

where
$$R_{\infty L} = \frac{\rho_\infty u_\infty L}{\mu_\infty}, \quad P = \frac{\mu_\infty c_{p\infty}}{k_\infty},$$

subscripts denote differentiation, and P Prandtl number. The velocity components u and v have been divided by the free stream velocity u_∞, while the pressure p_e, density ρ, enthalpy h and viscosity μ have been divided by their free stream values. M_∞ is the free stream Mach number and γ the specific heat ratio.

One then eliminates the density derivatives through the use of the state equation (3.4) and rewrites the continuity equation (3.1) as

$$\rho u_x + \frac{u}{h}\frac{dp_e}{dx} - \frac{\rho u}{h} h_x = -\rho v_y + \frac{\rho v}{h} h_y. \tag{3.5}$$

For a given set of u and h profiles, (3.2), (3.3) and (3.5) represent, at each value of y, three equations in the four unknowns dp_e/dx, $\partial u/\partial x$, $\partial h/\partial x$ and v. Since one of the unknowns dp_e/dx does not vary across the layer, a fourth independent relation is needed at only one value of the y-ordinate. This relation obtains from the description of the supersonic inviscid outer flow at the somewhat arbitrary location of the boundary-layer outer edge, $y = \delta$. In the absence of incident waves, this description is given by the Prandtl–Meyer simple wave relation, whereas if both incident and reflected waves are important the more general description of characteristics for rotational flows is required. Except at singular locations, such as the throat or 'critical' station, the above-mentioned four relations suffice to determine uniquely v and the three unknown x-derivatives at all values of y in the viscous layer, including its upper boundary δ. The arbitrariness of δ, and the sensitivity of the results to its definition, are an important weakness of the theory which we shall investigate further in § 5.

Eliminating u_x and h_x in (3.5) in favour of dp_e/dx through the use of (3.2) and (3.3), after some manipulation one finds

$$\frac{uv_y - vu_y}{u^2} = -\frac{M^2-1}{\gamma p_e M^2}\frac{dp_e}{dx} - \frac{1}{p_e R_{\infty L}}\frac{h}{u}\left[\frac{(\mu u_y)_y}{u} - \frac{1}{P}\frac{(\mu h_y)_y}{h} - (\gamma-1)M_\infty^2\frac{\mu u_y^2}{h}\right]. \tag{3.6}$$

One notes that the left-hand side of this last result is simply the exact partial differential $\partial/\partial y(v/u)$. Integrating (3.6) across the viscous layer, applying the boundary condition $v = 0$ at $y = 0$, and rearranging terms, one obtains

$$\frac{dp}{dx}\int_0^\delta \frac{M^2-1}{\gamma p_e M^2}\,dy = -\tan\theta_e - \frac{1}{p_e R_{\infty L}} \times \int_0^\delta \frac{h}{u}\left[\frac{(\mu u_y)_y}{u} - \frac{1}{P}\frac{(\mu h_y)_y}{h} - (\gamma-1)M_\infty^2\frac{\mu u_y^2}{h}\right]dy, \tag{3.7}$$

where $\tan\theta_e = v_e/u_e$ is the flow deflexion of the streamline at the outer edge of the boundary layer. The integration of (3.6) could also be terminated at any intermediate value of y, in which case

$$v = -\frac{u}{p_e}\frac{dp_e}{dx}\int_0^y \frac{M^2-1}{\gamma M^2}\,dy - \frac{u}{p_e R_{\infty L}} \times \int_0^y \frac{h}{u}\left[\frac{(\mu u_y)_y}{u} - \frac{1}{P}\frac{(\mu h_y)_y}{h} - (\gamma-1)M_\infty^2\frac{\mu u_y^2}{h}\right]dy. \tag{3.8}$$

Equation (3.8) illustrates a point that should be kept in mind in boundary-layer analyses. One is never free to specify arbitrarily both v and dp_e/dx in initiating a boundary-layer calculation for a prescribed set of u and h profiles. If dp_e/dx is specified, the v profile is completely determined from (3.8). If the choice of dp_e/dx is left open, one is free to specify v at, at most, a single off-centreline position, say the value of v at the boundary-layer outer edge. This value v_e then determines dp_e/dx, and hence the rest of the v profile.

Although (3.7) involves integrals across the layer, a distinction should be drawn between this result, derived without approximation from the boundary-layer equations, and the equivalent result that obtains from the averaging methods of multimoment integral theory. In the latter theory each of the boundary-layer conservation equations and the velocity moments of these equations is integrated across the viscous region. Thus, the detailed dynamic behaviour that is described by the locally valid expressions (3.2) and (3.3) for the x-derivatives of u and h is averaged out, and expressions for x-derivatives of integral properties of u and h across the layer are obtained instead. In contrast, in the present theory only the continuity equation (3.5) is integrated across the layer, and then only after one has substituted the exact local expressions for u_x and h_x from boundary-layer theory into the continuity equation. Thus, neither is there an averaging of the momentum and energy transport processes occurring in the layer, nor do any x-derivatives of integral properties appear. As we shall see in the next subsection, the integrals in (3.7) represent the cumulative contributions of pressure and diffusive mechanisms in each streamtube to the displacement of the outer flow.

(i) *The pressure-area relation*

Equation (3.7) is the exact pressure-area relation for a two-dimensional boundary layer. Comparison with (2.1) shows that the analogue of dA/dx, the slope of the quasi one-dimensional streamtube wall, is $\tan\theta_e$, the inclination of the outer edge streamline. There are two contributions to $\tan\theta_e$: an inviscid contribution which relates the pressure gradient in the x-direction to the adiabatic expansion of the integrated streamtube area, and a viscous contribution which represents an integrated streamline displacement effect due to the diffusion of vorticity and thermal energy. The behaviour of the inviscid contribution is directly analogous to the inviscid quasi one-dimensional streamtube. One observes that if $dp_e/dx > 0$, the inviscid pressure field produces a positive or negative contribution to $\tan\theta_e$ depending on whether

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy \quad \text{is} \quad < \text{ or } > 0,$$

in that order, while the reverse is true if $dp_e/dx < 0$. If the contribution of the viscous term to $\tan\theta_e$ is neglected in (3.7), the outer edge streamline diverges or converges according to whether the

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy \quad > \text{ or } < 0,$$

in exactly the same manner as the inviscid quasi one-dimensional streamtube does according to whether $M >$ or < 1. The integral

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy$$

effectively represents a continuous distribution of infinitesimal inviscid streamtubes each of thickness dy whose overall thickness is δ. This integral is not new, having appeared earlier in Lighthill's (1950, 1953*a*, *b*) linearized theory of upstream influence in a supersonic boundary layer, where it is related to the logarithmic decrement of upstream influence.

It is natural, in view of the preceding discussion, to define the throat station for a two-dimensional viscous layer to be that station for which a finite pressure gradient produces no change in the flow inclination of the streamline at the outer edge of the boundary layer, i.e. where

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy = 0. \tag{3.9}$$

Unlike the one-dimensional case, where the streamtube boundaries are parallel at the sonic throat, $\tan\theta_e \neq 0$ at the throat station of a viscous layer as we have defined it above, but has some positive or negative background value given by the viscous integral term in (3.7). Equation (3.9) is more general: it applies to adiabatic and non-adiabatic flows of arbitrary Prandtl number, and is far simpler than the equivalent expression that emerges from the approximate integral techniques (e.g. compare with equation (3.6) in Reeves & Lees 1965).

As a prelude to the more detailed discussion in §4 and 5 of the behaviour of the integral in (4.5), we note here several qualitative features. The defining integral for the throat does not depend on the u and h profiles separately but on a single combined variable, the Mach number. The integral does not involve θ_e directly, and hence the equation for the outer flow. Equation (3.9) can be solved for an arbitrary value of M_e by varying the Mach number profiles in the viscous region. In general, M_e will be a function of δ and any parameters α_i, $i = 1, 2, 3, \ldots, n$ used to describe the Mach number profiles in the viscous layer. The integrand increases monotonically from $-\infty$ at $M = 0$ to 1 at $M = \infty$. Thus, the integrand is singular for boundary-layer flows which satisfy a zero-slip boundary condition. These flows require special treatment. The major contribution to the integral accrues from the low-speed flow in the subsonic portion of the viscous layer. Only the case of free shear layers is examined here. For these flows, where $M_e > 1$ and the centreline Mach number M_c is variable and subsonic, a throat will, in general, be present.

(ii) *The throat constraint*

One observes that, if dp_e/dx is to be bounded at the station where (3.9) is valid, then from (3.7)

$$\tan\theta_e^* = -\frac{1}{p_e^* R_{\infty L}}\int_0^\delta \frac{h}{u}\left[\frac{(\mu u_y)_y}{u} - \frac{1}{P}\frac{(\mu h_y)_y}{h} - (\gamma-1)\,M_\infty^2\frac{\mu u_y^2}{h}\right]dy. \qquad (3.10)$$

The star superscript is used to denote conditions in the outer inviscid flow at the throat station. Since the integral in (3.10) need not vanish, the throat station for a two-dimensional viscous layer will not, in general, be a minimum area station.

The description of the viscous inner flow, represented by the integral term in (3.10), is directly coupled to the description of the inviscid outer flow through the latter's relation to $\tan\theta_e$. Thus (3.10) is a compatibility constraint between the motion of the inviscid stream and the integrated behaviour of the viscous stream. This constraint depends on only the flow variables at the throat station, not on their forward x-derivatives. In contrast, at all stations other than the throat station, the motion of the inner viscous stream can always adjust to that of the outer stream due to the coupling between the streamline displacements of the viscous layer and the induced pressure gradient, dp_e/dx that this displacement generates. That being the case, there is never any constraint on the flow variables themselves. Physically speaking, there is a partial breakdown in communication between the two fluid streams at the throat station such that the inner viscous stream may not be able to fill the boundaries created for it by the motion of the outer flow.

To illustrate the mathematical nature of the constraint imposed by (3.10), consider the case where incident waves are not important, and $\tan\theta_e$ is described by the Prandtl–Meyer simple wave relation

$$\theta_e = \left[\frac{\gamma+1}{\gamma-1}\right]^{\frac{1}{2}}\left(\tan^{-1}\left[\frac{\gamma-1}{\gamma+1}(M_\infty^2-1)\right]^{\frac{1}{2}} - \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}(M_e^2-1)\right]^{\frac{1}{2}}\right) + \tan^{-1}[M_\infty^2-1]^{\frac{1}{2}} - \tan^{-1}[M_e^2-1]^{\frac{1}{2}} = \nu(M_\infty)-\nu(M_e). \quad (3.11)$$

Here $\nu(M)$ is the Prandtl–Meyer function and the reference station is chosen so that at downstream infinity the outer flow is once again parallel to the axis, $\theta_e = 0$, and the free stream pressure p_∞ and Mach number M_∞ recovered. From (3.9) and (3.11), (3.10) will take the form $F(M_\infty, M_e^*, \delta, \alpha_i) = 0$, where $i = 1, 2, \ldots, n-1$. Note that the i indices stop at $n-1$, since any one of the parameters α_i, or for that matter δ, can in principal be eliminated by the use of (3.9). For known free stream conditions, (3.10) provides a compatibility relation at the throat station between M_e^*, $\delta(x^*)$ and any $n-1$ parameters used in the description of the viscous-layer profiles. It is evident that only for $n = 1$, or when a one-parameter family of profiles is used to describe the inner flow, can there be a unique relation between the inner and outer fluid streams. In this case, for given values of M_∞ and $\delta(x^*)$, (3.10) reduces to a transcendental equation for M_e^*, whose real eigenvalues will comprise a discrete set, should they exist. Since each eigenvalue of M_e^* is related to the free parameter in the inner profiles by (3.9), the latter will also be determined. For all other cases where $n > 1$, the throat constraint, by itself, can never determine a unique inner and outer flow, but simply provides a single independent relation between M_e^*, $\delta(x^*)$ and the various α_i.

(iii) *The pressure gradient at the throat station*

At any station, the pressure gradient required to produce the flow inclination $\tan\theta_e$ of the streamline at the boundary-layer outer edge is, from (3.7),

$$\frac{dp_e}{dx} = \frac{-p_e\tan\theta_e + \dfrac{1}{R_{\infty L}}\displaystyle\int_0^\delta B\,dy}{\displaystyle\int_0^\delta \frac{M^2-1}{\gamma M^2}\,dy} = -\frac{N(x)}{D(x)}, \quad (3.12)$$

where

$$B = \frac{h}{u}\left[\frac{(\mu u_y)_y}{u} - \frac{1}{P}\frac{(\mu h_y)_y}{h} - (\gamma-1)\,M_\infty^2\frac{\mu u_y^2}{h}\right]. \quad (3.13)$$

The numerator N and denominator D are functions of x only because of the y integration. At the throat station, dp_e/dx is of 0/0 form, since $D = 0$ by (3.9) and $N = 0$ by (3.10). This indeterminate form can have more than one solution and is evaluated through a somewhat elaborate application of L'Hopital's rule, which is in principle equivalent to the derivation of (2.4). In the limit notation,

$$\frac{dp_e(x^*)}{dx} = \lim_{x\to x^*} -\left(\frac{dN}{dx}\bigg/\frac{dD}{dx}\right),$$

where x^* denotes the throat station. This limit analysis follows.

Applying Leibnitz's rule for differentiation of an integral, and using the defining relation for the Mach number, $M^2 = M_\infty^2(u^2/h)$, to express M_x in terms of u_x and h_x, one obtains for the derivative of the denominator in (3.12):

$$\frac{dD}{dx} = \int_0^\delta \frac{2}{\gamma M^2}\left(\frac{u_x}{u} - \frac{h_x}{2h}\right) dy + \frac{M_e^2 - 1}{\gamma M_e^2}\frac{d\delta}{dx}. \tag{3.14}$$

Similarly, for the derivative of the numerator, one has

$$\frac{dN}{dx} = \frac{dp_e}{dx}\tan\theta_e + p_e \sec^2\theta_e \frac{d\theta_e}{dx} + \frac{1}{R_{\infty L}}\int_0^\delta \frac{\partial B}{\partial x} dy, \tag{3.15}$$

where we have taken advantage of the fact that $B \to 0$ as $y \to \delta$. To determine dN/dx and dD/dx in terms of the u and h profiles and the edge conditions, one needs (*a*) to eliminate v, which appears implicitly in (3.14) and (3.15), and the unknown x-derivatives u_x and h_x in favour of dp_e/dx; (*b*) to evaluate $\partial B/\partial x$, which contains troublesome cross derivative terms of the type u_{xy}, u_{xyy}, h_{xy} and h_{xyy}; and (*c*) to express $d\delta/dx$ in terms of known quantities.

Consider first (*a*) above. v can be eliminated from the boundary-layer momentum and energy equations (3.2) and (3.3) by the use of equation (3.8). The desired expressions for u_x and h_x are

$$u_x = U\frac{dp_e}{dx} + \overline{U}, \tag{3.16}$$

$$U(x,y) = -\frac{1}{\gamma\rho u M_\infty^2} + \frac{u_y}{\gamma p_e}\int_0^y \frac{M^2-1}{M^2} dy, \tag{3.16a}$$

$$\overline{U}(x,y) = \frac{1}{R_{\infty L}}\frac{(\mu u_y)_y}{\rho u} + \frac{u_y}{p_e R_{\infty L}}\int_0^y B\, dy, \tag{3.16b}$$

and

$$h_x = H\frac{dp_e}{dx} + \overline{H}, \tag{3.17}$$

$$H(x,y) = \frac{\gamma - 1}{\gamma\rho} + \frac{h_y}{\gamma p_e}\int_0^y \frac{M^2-1}{M^2} dy, \tag{3.17a}$$

$$\overline{H}(x,y) = \frac{1}{PR_{\infty L}}\frac{(\mu h_y)_y}{\rho u} + \frac{\gamma - 1}{R_{\infty L}} M_\infty^2 \frac{\mu u_y^2}{\rho u} + \frac{h_y}{R_{\infty L}}\int_0^y B\, dy. \tag{3.17b}$$

Note that the expressions for U, $\overline{U}$, H and $\overline{H}$ do not contain any x-derivatives or v and, as required, are functions only of the u and h profiles and the edge conditions.

Ordinarily, one might expect that the cross-derivative terms mentioned in (*b*) in the penultimate paragraph would pose a major source of difficulty. However, in view of (3.16) and (3.17), these cross derivatives, too, can be expressed solely in terms of dp_e/dx and the profile shapes:

$$\left.\begin{aligned} u_{xy} &= U_y\frac{dp_e}{dx} + \overline{U}_y, & h_{xy} &= H_y\frac{dp_e}{dx} + \overline{H}_y, \\ u_{xyy} &= U_{yy}\frac{dp_e}{dx} + \overline{U}_{yy}, & h_{xyy} &= H_{yy}\frac{dp_e}{dx} + \overline{H}_{yy}. \end{aligned}\right\} \tag{3.18}$$

Assuming a viscosity law in which μ is a function of only temperature, one can write $\partial B/\partial x$ with the aid of (3.18) as

$$\frac{\partial B}{\partial x} = B_1 \frac{dp_e}{dx} + B_2, \tag{3.19}$$

after some algebra. B_1 and B_2 are given by (A 1) and (A 2) in the appendix, and are functions of only the u and h profiles, their y-derivatives and the edge conditions.

Referring to (*c*), an independent expression for $d\delta/dx$ is somewhat arbitrary, since δ itself is not clearly defined. However, a reasonable representation for $d\delta/dx$, for present purposes, can be obtained by requiring that $d\delta/dx = d\bar{\delta}/dx$, where $\bar{\delta}$ is the viscous-layer displacement thickness,

$$\bar{\delta} = \int_0^{\delta} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy. \tag{3.20}$$

From (3.1) the overall continuity equation can be written

$$\frac{d\delta}{dx} = \tan\theta_e + \frac{1}{\rho_e u_e} \frac{d}{dx} \int_0^{\delta} \rho u \, dy. \tag{3.21}$$

Differentiating (3.20), and equating this result to $d\delta/dx$ in (3.21), one obtains

$$\frac{d\delta}{dx} = \tan\theta_e + \frac{\delta - \bar{\delta}}{\rho_e u_e} \frac{d(\rho_e u_e)}{dx}.$$

$d\rho_e/dx$ and du_e/dx can be expressed in terms of dp_e/dx from the equation of state and the isentropic relations of the outer flow:

$$\frac{du_e}{dx} = -\frac{dp_e/dx}{\gamma \rho_e u_e M_\infty^2} \left(\frac{1 + \phi_e [M_e^2 - 1]^{\frac{1}{2}}}{1 + \phi_e^2}\right), \tag{3.22}$$

$$\frac{dh_e}{dx} = -(\gamma - 1) M_\infty^2 u_e \frac{du_e}{dx}, \tag{2.23}$$

where $\phi_e = \tan\theta_e$ and is given by (3.11). Thus, a suitable expression for $d\delta/dx$ is

$$\frac{d\delta}{dx} = \phi_e + \frac{dp_e}{dx}(\delta - \bar{\delta}) \left[\frac{M_e^2 - 1 + \gamma(M_e\phi_e)^2 - [M_e^2 - 1]^{\frac{1}{2}}(\phi_e + (\gamma - 1) M_e^2)}{\gamma p_e M_e^2 (1 + \phi_e^2)}\right]. \tag{3.24}$$

One is now ready to write dN/dx and dD/dx in the desired form, in which v, and all the x-derivatives appearing in (3.14) and (3.15), are expressed in terms of dp_e/dx. Combining (3.14), (3.16), (3.17) and (3.24) for dD/dx, and (3.15), (3.16), (3.17) and (3.19) for dN/dx, one obtains

$$\frac{dD}{dx} = \frac{M_e^2 - 1}{\gamma M_e^2} \phi_e + \int_0^{\delta} \frac{1}{\gamma M^2} \left(\frac{2\bar{U}}{u} - \frac{\bar{H}}{h}\right) dy + \frac{dp_e}{dx} \left[\int_0^{\delta} \frac{1}{\gamma M^2} \left(\frac{2U}{u} - \frac{H}{h}\right) dy \right.$$
$$\left. + (\delta - \bar{\delta}) \frac{M_e^2 - 1}{\gamma M_e^2} \frac{M_e^2 - 1 + \gamma(M_e\phi_e)^2 - [M_e^2 - 1]^{\frac{1}{2}}(\phi_e + (\gamma - 1) M_e^2)}{\gamma p_e M_e^2 (1 + \phi_e^2)}\right], \tag{3.25}$$

and

$$\frac{dN}{dx} = \frac{1}{R_{\infty L}}\int_0^\delta B_2\,dy + \frac{dp_e}{dx}\left[\phi_e + \left(\frac{[M_e^2-1]^{\frac{1}{2}}}{\gamma M_e^2}\right)(1+\phi_e^2) + \frac{1}{R_{\infty L}}\int_0^\delta B_1\,dy\right], \quad (3.26)$$

respectively.

Returning to (3.12), and evaluating (3.25) and (3.26) at $x = x^*$, one finds that the limiting expression for dp_e/dx at the throat station reduces to

$$\frac{dp_e(x^*)}{dx} = \lim_{x\to x^*} -\frac{dN/dx}{dD/dx} = -\frac{N_1(x^*)\,(dp_e/dx) + N_2(x^*)}{D_1(x^*)(dp_e/dx) + D_2(x^*)}. \quad (3.27)$$

The N_i and D_i $(i = 1, 2)$ depend on only the u and h profiles and their edge values, and are given in the appendix, (A 3) through (A 6). Equation (3.27) is a quadratic yielding two solutions for dp_e/dx at $x = x^*$.

$$\frac{dp_e(x^*)}{dx} = -\frac{(N_1+D_2) \pm [(N_1+D_2)^2 - 4D_1N_2]^{\frac{1}{2}}}{2D_1}. \quad (3.28)$$

If $dp_e(x^*)/dx$ is to be real, $(N_1+D_2)^2 > 4D_1N_2$. We shall see in § 5 that the positive and negative radicals lead to flows in which the

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy \quad \text{is} \quad <0 \text{ or } >0$$

downstream of the throat, in that order. Unlike the quasi one-dimensional streamtube, the magnitudes of the 'subcritical' and 'supercritical' roots for dp_e/dx at $x = x^*$ are not equal, except for the special case $N_1 + D_2 = 0$. When the radical vanishes, the two roots merge,

$$\frac{d}{dx}\int_0^\delta \frac{M^2-1}{M^2}\,dy = 0 \quad \text{at} \quad x = x^*,$$

and the throat has a finite length in the physical plane. Only at $x = x^*$ can a downstream branching be initiated. In general, for each set of u and h profiles satisfying (3.9) and (3.10) there will be two possible downstream solutions commencing at the throat station, provided real roots to (3.28) exist. In contrast, flows which do not contain a throat have a unique downstream behaviour.

(iv) *Upstream influence*

By analogy with the quasi one-dimensional streamtube, one expects to find some change in the upstream influence features of the flow in the vicinity of the throat station. The results of the studies of the interaction between an inviscid outer supersonic stream and an inner viscous stream, cited in § 1, all show that singular behaviour in the solutions developed in the neighbourhood of the throat. It was found, moreover, that only one particular value of some parameter left free in the initial data would permit a solution to develop downstream that was non-divergent as the throat station was approached. These results suggest that the problem was not well posed mathematically as an initial value problem. One might postulate that some kind of upstream influence might exist in such problems which prevents a proper initial value problem from appearing.

A distinctive feature of such flows is that they involve interaction with an outer stream and depend on this interaction to set the pressure gradient impressed on the viscous layer. In contrast, ordinary viscous flows of the boundary-layer type, which lack interaction with an outer stream, have prescribed pressure gradients aud are known to constitute well-posed initial value problems. However, in interaction problems, rather than taking the simple role of a driving term, the pressure gradient depends in a complicated way on the structures of both the boundary-layer and outer flows. It is this difference which accounts for the fact that the behaviour of interaction problems (treated as initial value problems) is different from that of those without interaction.

The actual dependence of the pressure field on the rest of the flow field will be quite complicated, in general. However, since the manner in which the pressure gradient does involve other variables is of such importance, it is appropriate here to review a recent analysis by Garvine (1968) which examined a simplified and linearized version of the system (3.1)–(3.4).

The object in that study was to examine the mathematical consequence of interaction. In order to proceed analytically, the complications arising from compressibility were omitted and the non-linear convection terms in the momentum equation replaced by a linear term involving u_x after the fashion of Oseen's linearization. A linearized variation of (3.11) was selected to describe the interaction, a form that would be valid in the event that an outer flow governed by the linearized supersonic flow equation prevailed. This linear set of model equations then appeared as

$$u_x + v_y = 0, \tag{3.29}$$

$$\overline{\rho u} u_x + \frac{p_{ex}}{\gamma M_\infty^2} = \frac{\bar{\mu}}{R_{\infty L}} u_{yy}, \tag{3.30}$$

$$p_{ex} = \kappa v_{ex}. \tag{3.31}$$

Equation (3.30) displays the linearized convection term with its constant coefficient $\overline{u\rho}$, a value that might be taken as average in the boundary layer for best application to a real flow. Similarly, $\bar{\mu}$ is an average viscosity coefficient in the layer.

Equation (3.31) expresses the fact that the pressure gradient is proportional to v_{ex}. In the event of linearized supersonic outer flow, the constant is

$$\kappa = \gamma M_\infty^2 (M_\infty^2 - 1)^{-\frac{1}{2}}.$$

One wishes to express p_{ex} in the momentum equation in terms of other dependent variables. From the continuity, (3.29), we may integrate once over y to find

$$v_e = -\int_0^\delta u_x \, dy.$$

Here, the boundary-layer outer edge where $y = \delta$ is taken as a constant for simplicity. Then

$$\frac{dp_e}{dx} = -\kappa \int_0^\delta u_{xx} \, dy. \tag{3.32}$$

Equation (3.32) enables one to replace dp_e/dx in the momentum equation and obtain an equation in terms of u alone, a linear integro-partial differential equation. The surprising feature of (3.32), however, is the fact that it involves an integral over the *second* derivative of u with respect to x. Such derivatives were among those dropped from the Navier–Stokes stress terms in making the boundary-layer approximation. In general, they are acknowledged to contain the upstream influence found in the Navier–Stokes equations. It is the interaction in this case that brings such a derivative back into the problem, and one may suspect some upstream influence along with it.

The solution for u appropriate to a wake flow can be developed from Garvine (1968) by means of a Laplace transform. Upon inversion one obtains the form

$$u = 1 + \sum_{k=1}^{\infty} \frac{\eta(y; s_k)}{D'(s_k)} e^{s_k x} + IF(y)\, e^{s_p x}. \tag{3.33}$$

The s_k constitute a set of real negative numbers, while s_p is a positive number. Thus, the first term above is required to satisfy the y boundary conditions, the second represents a decay of the initial profile, $u_i(y)$, and the third represents a divergent term. The sign of the divergent term depends on the factor I, so named because it depends only on the initial profile $u_i(y)$ and v_e at $x = 0$.

One may notice in these results a strong parallel with those reported concerning forward integration of the full boundary-layer equations with interaction. In both problems the downstream solution is divergent. Only for one value of a parameter in the intial data ($I = 0$ in the simplified problem) will a non-divergent solution obtain. Setting $I = 0$ imposes a constraint on the initial data and requires that $v_e(0)$ depend on the details of $u_i(y)$ and integrals over it.

The parallel strongly suggests that the system (3.1)–(3.4) also possesses upstream influence and thus cannot be attacked as an initial value problem. The results shown in (3.33) for the model set of equations show that if that problem is to be attacked as an initial value problem, then a constraint must be placed on the initial data.

Such would appear to be the case for the non-linear compressible system as well. In fact, an expression for dp_e/dx, similar to that of (3.32) was obtained in Weinbaum & Garvine (1968):

$$p_{ex} = \frac{-\dfrac{h_e}{ve}\displaystyle\int_0^{\delta} \alpha u_{xx}\, dy + T(u_x^2, u_x, u, p, h)}{1 + Z(u_x, u, p, h)}. \tag{3.34}$$

T and Z are complicated functions, but involve only the x-derivative u_x and its square. The coefficient α is defined as

$$\alpha \equiv \rho \frac{M^2 - 1 - M^2\left(\dfrac{h_y}{h} - \dfrac{u_y}{u}\right)\displaystyle\int_0^y \frac{M^2-1}{M^2}\, dy}{M^2 \dfrac{u_y}{u}\displaystyle\int_0^y \frac{M^2-1}{M^2}\, dy - 1}.$$

For α, h_e, v_e and Z constant and $T = 0$, (3.34) is formally identical to (3.31).

In the exact system, the upstream influence enters in a much more complicated role with the Mach number profiles playing an important part.

Somewhat more insight into the role played by the Mach number profiles may be seen by simplifying equations (3.2) and (3.3) by omitting the convection terms $\rho v u_y$ and $\rho v h_y$, respectively, but leaving all else intact. Then, the coefficient α reduces to

$$\alpha = \rho(1-M^2).$$

Thus, when the Mach number profiles shift from subsonic to supersonic in some average sense, a change in sign occurs in the term involving u_{xx}. If we examine what happens to the results for the linearized model set when κ is changed from a positive to a negative value (as in the case of physical interest), it is readily shown that s_p in (3.33) changes from positive to negative, and the divergent character of the solution vanishes. These reflexions bear out the suggestion that near the throat station the shift from subcritical to supercritical (subsonic to supersonic in the mean) flow eliminates the upstream influence from the problem, and hence also the divergent behaviour. Downstream of that station, then, the problem should be well set as an initial value problem.

The above developments do not, however, prove that precisely at the throat station the upstream influence should cease as one moves from the side of the throat where

$$\int_0^\delta (M^2-1)/M^2\,dy < 0$$

to the side where this integral is positive, but two arguments may be advanced in support of this. First, the numerical results described in § 5, where the equations were integrated in the downstream direction starting at the throat station, encountered no behaviour downstream suggestive in any way of upstream influence. Secondly, Lighthill (1953*a*, *b*) shows that, to a first approximation, the distance upstream, over which a small disturbance to a supersonic viscous layer is felt, is proportional to

$$-\int_0^\delta (M^2-1)/M^2\,dy$$

(so long as the integral itself is negative, as with a body boundary layer). This upstream influence then vanishes when the integral itself vanishes at the throat station.

The analogy of the two-dimensional viscous layer to the quasi one-dimensional streamtube is thus valid with respect to upstream influence as well. In both cases, the upstream influence appears to vanish downstream of the throat station as defined for each flow.

4. Transformation of basic relations and the selection of throat profiles

It is convenient in applying (3.9) and (3.10) to profiles typical of either boundary layers or free shear layers to define the new variables,

$$V = \frac{u}{u_e}, \quad T = \frac{h}{h_e}, \tag{4.1}$$

which are independent of the particular edge values u_e and h_e. Similarly, one wishes to define a scaled normal co-ordinate, which is independent of the actual physical thickness δ of the viscous layer, and is proportional to the Howarth–Dorodnitsyn variable of compressible boundary-layer theory, used to reduce the compressible equations to their incompressible form. Thus, we introduce the normalized density transformation

$$d\frac{y}{\delta} = h d\eta \bigg/ \int_0^{\eta_\delta} h d\eta. \tag{4.2}$$

Here, η_δ is the value of η at $y/\delta = 1$. In these new variables (3.9) and (3.10) can be rewritten as

$$\int_0^{\eta_\delta} \frac{M_e^{*2}(V^2/T)-1}{V^2} T^2 d\eta = 0, \tag{4.3}$$

and

$$R_{\infty L}\delta(x^*) = -\frac{k}{\tan\theta_e^*}\frac{h_e^{*\frac{3}{2}}}{p_e^*}\frac{M_\infty}{M_e^*}\int_0^{\eta_\delta} T d\eta \int_0^{\eta_\delta}\frac{T}{V}\left[\frac{V_{\eta\eta}}{V} - \frac{1}{P}\frac{T_{\eta\eta}}{T} - (\gamma-1)M_e^{*2}V_\eta^2\right] d\eta, \tag{4.4}$$

where we have used the Chapman–Rubesin viscosity law and k is the Chapman–Rubesin constant. The edge conditions are given by (3.11) and the other isentropic relations for the outer flow. Pursuing the discussion at the end of parts (i) and (ii) in § 3 one sees that (4.3) and (4.4) are functional relationships of the form

$$F(M_e^*, \eta_\delta, \alpha_i) = 0, \quad i = 1, 2, \ldots, n, \tag{4.5a}$$

$$R_{\infty L}\delta = R_{\infty L}\delta(M_e^*, M_\infty, \eta_\delta, \alpha_i), \quad i = 1, 2, \ldots, n-1, \tag{4.5b}$$

where the α_i are the parameters describing the V and T profiles, and one of the α_i in (4.5*b*) has been eliminated through the use of (4.5*a*).

For an arbitrary but constant value of $R_{\infty L}\delta(x^*)$, the Reynolds number scaling for the pressure gradient at the throat station (3.28) can be deduced in the following manner. With $R_{\infty L}\delta$ held constant, $dy \propto (1/R_{\infty L})\, d\eta$. From (A 3) through (A 6) one can show that $D_1 \propto (1/R_{\infty L})$, $D_2 \propto 1$, $N_1 \propto 1$ and $N_2 \propto R_{\infty L}$. Thus, for a given value of $R_{\infty L}\delta$ at the throat, $dp_e(x^*)/dx$ is proportional to $R_{\infty L}$. Therefore, $dp_e(x^*)/dx$ can be calculated for a unit free stream Reynolds number, and scaled to any other $R_{\infty L}$, simply by multiplying by $R_{\infty L}$. Pressure gradient curves, like those shown in figure 5, apply for all $R_{\infty L}$.

Because of the saddle-point character of the throat singularity, and the apparent change in the upstream influence properties of the interaction process that occurs at the throat station, this station provides a convenient initial station for a downstream integration. Experience with the hypersonic wake problem shows that if the viscous-layer profiles are prescribed upstream of the wake throat, minute adjustments, involving double precision accuracy, of the free parameter used in the description of these profiles are required if one is to integrate numerically, starting at the wake reattachment stagnation point, a significant distance toward the wake throat. In general, such costly and delicate trial and error adjustments are avoided if one can start the solution directly at the throat station with a compatible set of u and h profiles and associated pressure gradient.

Since the pressure gradient at the throat can be determined for an arbitrary set of u and h profiles from (3.28), the remaining problem is the selection of a suitable set of throat profiles for the problem of physical interest. To aid in this selection process one has two conditions, (4.3) and (4.4), at one's disposal that must be satisfied. Beyond this, the accuracy of the throat profiles will depend largely upon how much is known about the upstream flow. If one wishes to avoid a detailed analysis of that flow, then this knowledge will be limited very likely to only certain gross features of the upstream motion; probably, the most important of these will be $R_{\infty L}\delta = R_{\infty\delta}$, the freestream Reynold number based on viscous-layer thickness at $x = x^*$. We shall see in § 5 that this gross feature is particularly convenient for wake-type flows where $\theta_e^* \ll 1$, since $R_{\infty\delta}$ does not have to be known accurately to obtain a reasonable approximation for the throat profile shapes. In general, the specification of $R_{\infty\delta}$ would appear to be the minimum additional information necessary to select a set of u and h throat profiles from a given one-parameter family. It is evident that, once an approximate value for $R_{\infty\delta}$ is obtained either from an appropriate free shear layer calculation or from experiment, the throat constraint (4.4) or (4.5b) will determine a discrete set of values for M_e^* or θ_e^*. For each such eigenvalue, the defining relation for the throat (4.3) or (4.5a) will then determine the free parameter α_1, which distinguishes the profile shapes. For flows in which M_e^* is not a sensitive function of $R_{\infty\delta}$ ($\theta_e \ll 1$), the accuracy of the throat profiles will depend primarily on how suitable the particular one-parameter family chosen is for the problem being studied. Of course, each new piece of information that goes into refining the u and h profiles (e.g. the centre-line stagnation enthalpy at the throat for non-adiabatic wakes) must first be obtained from an appropriate model of the upstream flow. The sensitivity of M_e^* in (4.5b) to each new input will then determine how sophisticated an upstream model is required to obtain this input. While a detailed solution of the upstream flow cannot be avoided if very accurate throat profiles are required, the throat analysis is still an essential part of such a detailed solution. In these detailed models—e.g. Weiss's (1967) model of the hypersonic near wake—one free parameter such as the base pressure or stagnation enthalpy must always be left open in the upstream description so as to satisfy the throat constraint (4.4) downstream.

Since the throat problem of greatest contemporary interest is the hypersonic near wake, the authors have chosen it to illustrate the theory. Other viscous throats may be present in jets, highly cooled boundary layers and other situations, but these applications will not be examined here. The selection of an accurate family of throat profiles for the high Reynolds number hypersonic wake is difficult because of the complexity of the flow in the wake recompression region (e.g. see the experimental measurements of Batt & Kubota 1968). Despite this uncertainty, most investigators seem to agree that the Stewartson (1954) family of wake-similar profiles provides a reasonable description of the laminar portion of the wake downstream of the reattachment stagnation point, and that these profiles become increasingly more accurate as one proceeds downstream.

The Stewartson wake-similar profiles are derived from the Falkner–Skan equation for compressible adiabatic flow:†

$$f_{\eta\eta\eta}+ff_{\eta\eta}+\beta(1-f_\eta^2) = 0, \tag{4.6}$$

and satisfy the boundary conditions appropriate for a free shear layer

$$\left.\begin{array}{l} \dfrac{\partial V}{\partial \eta} = \dfrac{\partial T}{\partial \eta} = 0, \quad \eta = 0, \\ \left.\begin{array}{l} V = 1, \quad T = 1, \\ V_\eta = T_\eta = 0, \end{array}\right\} \eta \to \infty. \end{array}\right\} \tag{4.7}$$

Here, $V = f_\eta$; V approaches unity in an exponential manner and is within 0·1 % of unity when η is roughly 4 or 5, depending on β. The pressure gradient parameter in the similarity theory, β, is a single valued function of the centre-line velocity ratio V_c; this function is plotted in Kennedy (1964). The centre-line Mach number M_c, the parameter we shall use to distinguish the profiles, is related to V_c by

$$M_c^2 = \frac{V_c^2}{\dfrac{\gamma-1}{2}(1-V_c^2)+\dfrac{1}{M_e^2}}. \tag{4.8}$$

In the present context the value of β is dissociated from its mathematical definition in the similarity theory and is used simply as a parameter to identify profiles compatible with the throat constraint. The h or T profile obeys the Crocco integral for adiabatic flow with $P = 1$,

$$T = 1+\left(\frac{\gamma-1}{2}\right)(1-V^2)\,M_e^{*2}, \tag{4.9}$$

and hence is determined for a given value of M_e^* once the boundary-value problem posed by (4.6) and (4.7) is solved for V.

5. Numerical results

In this section numerical results will be presented for the one-parameter family of wake-like profiles described by (4.6), (4.7) and (4.9). As discussed in the previous section, these profiles depend on only the parameter M_c and the external Mach number M_e^*; the latter is determined by the throat-constraint equation (4.4), which relates M_e^* and $R_{\infty\delta}$. Therefore, for a given value of M_e^*, (4.3) reduces to an integral equation for the parameter M_c, which is then solved subject to the auxiliary equations and conditions for the profiles (4.6), (4.7) and (4.9).

The above-mentioned solution is readily accomplished by trial and error numerical techniques. One chooses a trial value of M_c; the corresponding value of V_c is then obtained from (4.8); the boundary conditions for (4.6) are split between $\eta = 0$ and $\eta = \infty$. One finds that, for each trial initial value of $f_\eta(0)$ or V_c,

† Reeves & Lees (1965, p. 2094), state that the only similar solution for a compressible free shear layer is for adiabatic flow, in which case the Cohen–Reshotko (1956) equations for non-adiabatic flow reduce to (4.6) above.

there is only one value of β for which V will asymptotically approach unity as $\eta \to \infty$. A satisfactory first estimate for this value of β can be obtained from figure 4 in Kennedy (1964), and β can be determined to greater accuracy by numerical iteration. Once the V and T profiles are determined in this manner, the integral in (4.3) is evualated numerically, starting at $y = \eta = 0$ and terminating at the value of y, $y = Y(x)$, or equivalently the value of η for which the integral vanishes. In general, this value of η will differ from η_δ, the definition of the boundary-layer outer edge. The difference is then used as a guide for selecting the next trial value of M_c. Adopting a simple iterative scheme based on the aforementioned difference, the authors solved (4.3). M_c could be determined to four significant digits using less than 15 sec of IBM 7094 computer time.

Figure 1 shows the solution to (4.3) for M_c as a function of the location of the $y = Y$ surface in V space, with M_e^* treated as a fixed parameter. Equivalently, the curves show the sensitivity of the throat profiles, as indicated by the parameter M_c, to the definition of the boundary-layer outer edge (e.g. if $M_e^* = 1{\cdot}5$ and V_e is varied from $V_e = 0{\cdot}998$ to $V_e = 0{\cdot}98$, M_c changes from 0·705 to 0·760). The figure clearly disproves the suggestion by Weinbaum (1967) that the wake throat might be eliminated entirely by suitably redefining the boundary-layer outer edge. It is evident that the $y = Y$ surface moves quickly into the inviscid flow for all values of M_e^* as the centre-line Mach number M_c is decreased past 0·6. Thus, the boundary-layer outer edge must always intersect the $y = Y$ surface in an integration that is initiated at the wake rear stagnation point. This intersection then defines the wake throat.

A second point of interest in figure 1 is the change in behaviour of the $y = Y$ surface as the inviscid flow Mach number M_e^* is increased. At the low supersonic speeds (e.g. $M_e^* = 1{\cdot}5$) small changes in the definition of δ (say from the location where $V_e = 0{\cdot}998$ to $V_e = 0{\cdot}98$) lead to significant changes in M_c, and hence to significant changes in the throat profiles themselves. At high Mach numbers (e.g. $M_e = 10$) M_c is much more insensitive to the definition of the boundary-layer outer edge. One may expect such behaviour because of the well-established properties of hypersonic boundary layers. For increasing external Mach number the boundary-layer edge becomes 'sharper' and is less and less subject to arbitrary location.

The behaviour just discussed is also illustrated in figure 2, where the solution of the centre-line Mach number at the throat station M_c^* is shown for four different definitions of δ, $V_e = 0{\cdot}95$, 0·995, 0·9995 and 0·99995, which span nearly all of the values used in the literature. One notes that the difference between any two solution curves for M_c^* for these four definitions of δ continuously decreases as M_e^* increases, and appears to asymptote to a fixed difference as M_e^* approaches infinity. At $M_e^* = 20$, M_c^* differs by 1·63 % upon comparing the results for $V_e = 0{\cdot}95$ and 0·995, by 0·38 % for $V_e = 0{\cdot}995$ and 0·9995, and 0·23 % for $V_e = 0{\cdot}9995$ and 0·99995. For smaller values of M_e^* these percentage differences are all somewhat larger, but again the difference between any two neighbouring pairs of curves becomes smaller as larger values of V_e are considered in the definition of δ. Once $V_e = 0{\cdot}995$, a further increase in δ, such that $1 - V_e$ is decreased by two orders of magnitude, changes M_c by less than 1 % over most of the high Mach number

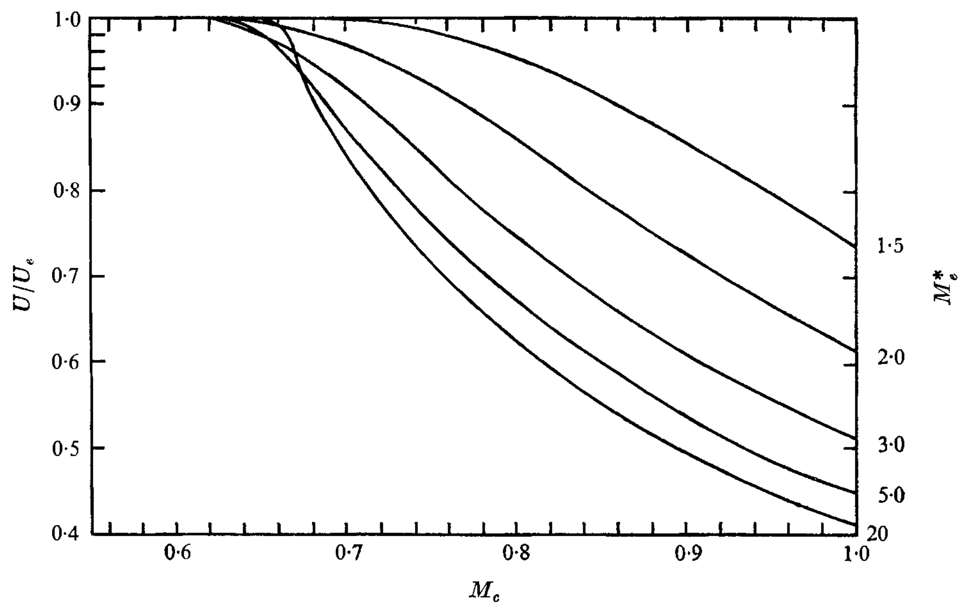

FIGURE 1. U/U_e *vs.* centre-line Mach number.

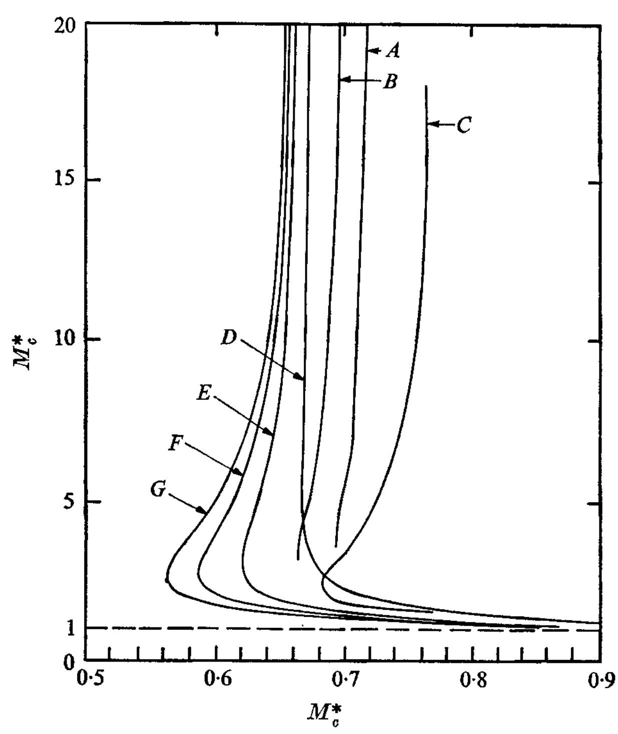

FIGURE 2. Edge Mach number *vs.* centre-line Mach number. *A*, Ai (1967), quartic profile. *B*, Ai (1967), cubic profile. *C*, Reeves & Lees (1965). *D*, Present theory, δ at $U/U_e = 0{\cdot}95$. *E*, 0·995. *F*, 0·9995. *G*, 0·99995.

range. As a practical matter, the sensitivity of the results to δ is unimportant, at least for the free shear layer.

What dependence there is on δ is a consequence of using boundary-layer equations in one part of the flow field, and inviscid flow equations in another. If boundary-layer equations are to be used, some arbitrary division of the flow field is unavoidable. The results must always be sensitive in some degree to that fraction of the flow field assigned to the boundary-layer equations. The key test is whether or not perturbations from some physically reasonable choice of δ affect the results to an unacceptable degree. The calculations show that for the problem considered, at least, they do not.

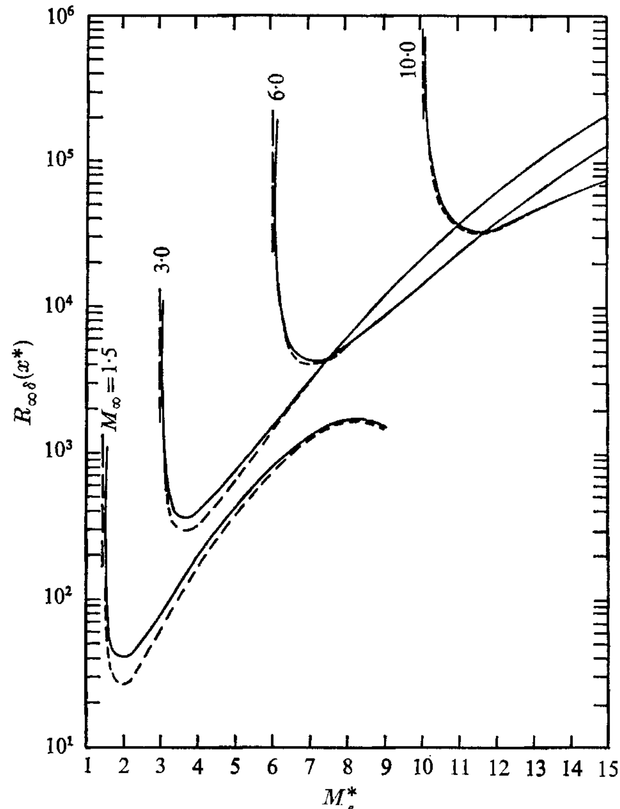

FIGURE 3. Reynolds number based on layer thickness at throat *vs.* edge Mach number. $y = \delta$ at $U/U_e = 0{\cdot}95$ ----; $y = \delta$ at $U/U_e = 0{\cdot}995$ ——. Vertical lines — — — solution of (5.4) for minima in $R_{\infty\delta}$ curves.

Also shown in figure 2 are the equivalent results of the integral methods (Ai 1967; Reeves & Lees 1965). Ai used cubic and quartic polynomical profiles, whereas Reeves & Lees employ the same Stewartson family used herein. Interestingly, the difference between Ai's solution curve and the present exact treatment of the boundary-layer equations is less than that between the Reeves & Lees solution curve and the present results, although the same profiles were used for the latter comparison. This observation suggests that the inaccuracy of the integral method due to the averaging of the governing equations is at least as important as the differences due to the differences in profile shape.

In figure 3 we have plotted $R_{\infty\delta}$ as a function of M_e^* from (4.4) with M_∞ as a fixed parameter. The approximately 200 cases calculated in plotting figures 3 (not all cases shown) and 4 required on the order of 20 min of IBM 7094 computer time. Figure 4 is an equivalent plot to figure 3, except that the inviscid flow inclination angle θ_e^* is used as the abscissa. These two figures are intended to provide a comprehensive map of the Stewartson wake-like profiles that will satisfy the throat constraint for a wide range of $R_{\infty\delta}$ and M_∞ of interest. Once

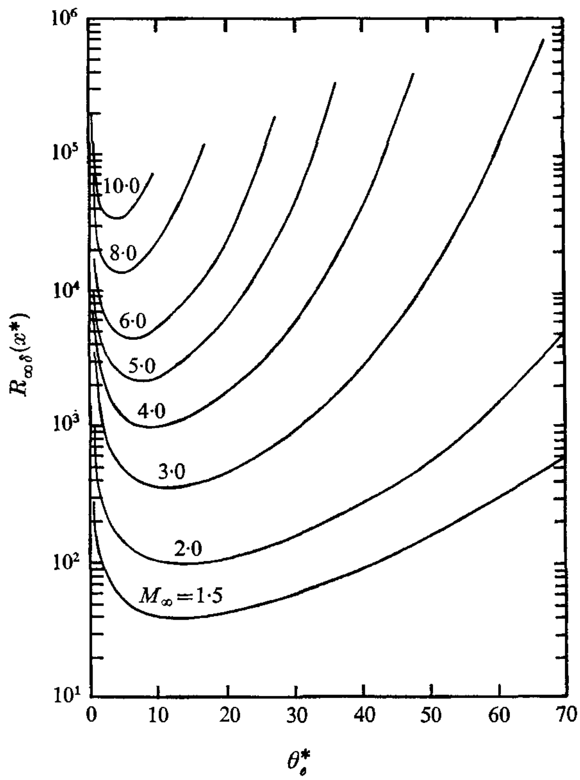

FIGURE 4. Reynolds number based on layer thickness at throat *vs.* edge streamline angle.

$R_{\infty\delta}$ has been estimated either by experiment or by an appropriate free shear layer calculation, figures 2, 3 or 4, and figure 5 (which gives the associated pressure gradient), provide all the necessary information needed to initiate a far wake calculation at the throat station. In general, $R_{\infty\delta}$ will depend upon M_∞ and the body geometry. Again, this simple reformulation of the initial value problem for the far wake is only possible if one uses a one-parameter family of initial profiles, since the throat constraint reduces only by one the number of free parameters used in the u and h profile descriptions.

Several points are of particular interest in figures 3 and 4. One notes from figure 3 that the so-called 'eigenvalues' for M_e^* are sensitive to the definition of δ for only the lower supersonic values of M_∞. If δ is taken to be at $V_e = 0{\cdot}95$ or $0{\cdot}995$, then for $M_\infty > 5$ there is no significant change in the $R_{\infty\delta}(x^*)$ curves. Zero, one, two or three eigenvalues may exist for M_e^*, depending on the choice of δ. However, as one observes from figure 5, not all portions of these $R_{\infty\delta}(x^*)$ curves are realizeable, since (3.28) for the pressure gradient does have real solutions for all values of $M_e^* > M_\infty$. Also, the physical significance of the highest eigenvalues is questionable, since figure 4 shows that in many cases they occur for flow deflexion angles in excess of the angles for which one would normally expect the boundary-layer equations to be valid.

Perhaps the most intriguing feature in figures 3 and 4, which was also reported by investigators using the integral method (Webb *et al.* 1965; and Ai 1967), is the failure of any throat solutions to exist when $R_{\infty\delta}$ is decreased below some minimum critical value. Conversely, one can state from figures 3 and 4 that for given values of M_∞ and $R_{\infty L}$ there is a minimum thickness of $\delta(x^*)$ for which a throat can exist. To discover the origin of this behaviour we return to (3.10) and rescale the y co-ordinate so that the upper limit on the integral is unity. Thus, one lets $dy = \delta d\xi$, and (3.10) becomes

$$p_e^* \tan\theta_e^* = -\frac{1}{R_{\infty L}\delta}\int_0^1 B_{sc}\,d\xi, \tag{5.1}$$

$$B_{sc} = \frac{h}{u}\left[\frac{(\mu u_\xi)_\xi}{u} - \frac{1}{P}\frac{(\mu h_\xi)_\xi}{h} - (\gamma-1)\,M_\infty^2\mu\frac{\mu_\xi^2}{h}\right].$$

Differentiating with respect to M_e^*, one has

$$\frac{d\,(p_e^*\tan\theta_e^*)}{dM_e^*} = -\frac{1}{\delta R_{\infty L}}\int_0^1 \frac{\partial B_{sc}}{\partial M_e^*}\,d\xi + \frac{1}{\delta^2 R_{\infty L}}\left(\frac{\partial\delta}{\partial M_e^*}\right)\int_0^1 B_{sc}\,d\xi. \tag{5.2}$$

The first term on the right-hand side of (5.2) is a slowly varying function of M_e^*, since M_c^* and consequently the profile shapes are not sensitive functions of M_e^* except as $M_e^* \to 1$ (see figure 2). Therefore, (5.2) is primarily a balance between the $p_e^*\tan\theta_e^*$ term and the term involving $\partial\delta/\partial M_e^*$. Thus, when δ is a minimum $\partial\delta/\partial M_e^* = 0$, and

$$\frac{d(p_e^*\tan\theta_e^*)}{dM_e^*} \approx 0. \tag{5.3}$$

One can readily show from the isentropic relation for the outer flow (3.11), and the related expression for p_e, that $p_e^*\tan\theta_e^*$ has a maximum when

$$\sin 2\theta_e^* = -\frac{2}{\gamma M_e^{*2}}[M_e^{*2}-1]^{\frac{1}{2}}. \tag{5.4}$$

Solutions to this transcendental equation are listed in table 1 and are shown by the dashed vertical lines in figure 4. The close agreement with the actual minima in the $R_{\infty\delta}$ curve is evident. These minima are a basically inviscid phenomenon. As $|\theta_e|$ is increased, p_e^* must decrease, since the amount of recompression required to turn the outer flow back parallel to the centre-line and recover the ambient pressure is increased.

M_∞	1·5	2·0	3·0	4·0	5·0	6·0	8·0	10·0	15·0	20·0
M_e^*	2·135	2·607	3·647	4·738	5·848	6·969	9·226	11·496	17·188	22·891

TABLE 1. Locus of minima for $R_{\infty\delta}$ curves

The solution curves for the pressure gradient per unit Reynolds number at the throat station that correspond to several of the M_∞ = constant curves in figures 3 and 4 are shown in figure 5. For given values of M_∞ and M_e^*, (3.28) need be calculated for only a single value of $R_{\infty L}$, since the throat pressure gradient can be scaled to any other $R_{\infty L}$ following the arguments presented in §4. Over

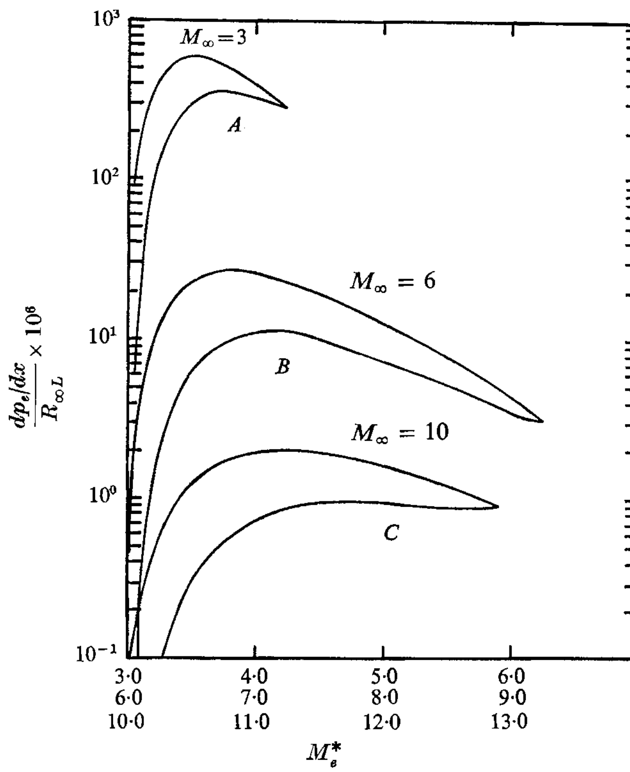

FIGURE 5. Throat pressure gradient *vs*. edge Mach number. *A*, upper scale. *B*, middle scale. *C*, lower scale.

most of the region of interest two roots exist, both positive. The lower root represents a flow where the centre-line Mach number accelerates through the throat, the integral

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy$$

becoming positive, whereas for the upper root the centre-line Mach number decelerates, and the integral becomes negative. The other interesting observation is that the two roots eventually merge and become one as M_e^* increases, and no solutions for dp_e/dx exist beyond this merger point. As stated previously, not all portions of the curves in figures 3 and 4 represent physically possible solutions. Each set of throat profiles must be associated with some real pressure gradient if it is to be physically meaningful.

Figure 6 illustrates the downstream behaviour starting at the throat station of representative profiles that satisfy the throat constraint (3.10) or (4.4). The initial conditions for the calculations were obtained from figure 2 and representative points on the $M_\infty = 6$ curve in figures 3 or 4, and 5. The compressible boundary-layer equations (3.1)–(3.4) were integrated numerically using an implicit finite difference scheme developed by Flügge-Lotz & Blottner (1962). The dashed curves in figure 6 correspond to the upper root for $dp_e(x^*)/dx$. For these solution curves the ambient pressure is not recovered, but a downstream stagnation point is achieved instead. These curves are terminated at the station where the axial velocity vanishes. These flows may be indicative of the presence of a downstream obstacle or a reverse jet.

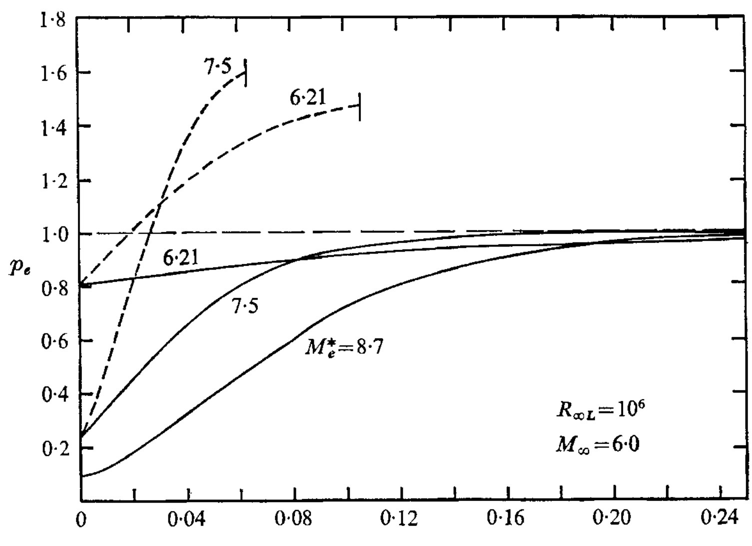

FIGURE 6. Pressure *vs.* distance from throat.

The solid curves in figure 6 are wake-like solutions in that they exhibit a recompression followed by a return to ambient conditions (note that x is expressed in terms of body length rather than base diameter). Again, the selection of the appropriate curve for a given body and prescribed set of flight conditions is tantamount to choosing the correct value for $R_{\infty\delta}$ in figures 3 or 4. Perhaps the most interesting feature of the wake-like curves in figure 6 is that somewhere between $M_e^* = 6{\cdot}21$ and $M_e^* = 7{\cdot}5$ there is a solution that minimizes the length of the pressure recovery region starting at $x = x^*$. This solution might represent the most efficient recompression process or, equivalently, the solution that leads to the minimum increase in entropy. This hypothesis is being studied further.

6. Conclusions

The more important general results of the investigation are listed below.

(i) A two-dimensional viscous layer satisfies an over-all pressure-area relation that is closely related to that of the one-dimensional adiabatic streamtube.

(ii) The behaviour of the viscous layer at the station where the

$$\int_0^\delta \frac{M^2-1}{M^2}\,dy = 0$$

is analogous to the one-dimensional sonic throat, in that (*a*) there is an equivalent change in the inviscid pressure-area behaviour for the layer treated as a whole, (*b*) there is a constraint present on the inclination angle of the outer edge streamtube, (*c*) dual solutions for the pressure gradient exist leading to either subsonic or supersonic behaviour in the mean downstream of the throat, and (*d*) there is a change in upstream influence properties of the flow at this station.

(iii) The linearized theory of Garvine (1968) points to the existence of a single growing mode for subcritical initial value problems formulated upstream of the viscous throat, and the disappearance of this mode as one crosses the throat station.

(iv) The defining expression for the throat relates the local Mach number M_e in the supersonic inviscid outer stream to the n parameters used in the profile descriptions of the inner viscous stream. The throat constraint relates $R_{\infty\delta}$ to M_e and $n-1$ of the above parameters. Only when $n = 1$, a one-parameter family of profiles, will zero or a discrete number of compatible inner and outer flow solutions exist for a given value of $R_{\infty\delta}$ and prescribed ambient conditions.

(v) For a one-parameter description, the throat provides a convenient initial station for a downstream integration. All the initial data can be related to a single quantity $R_{\infty\delta}$. This simplified description, for example, permits one to determine an approximate set of initial conditions for the downstream flow in the supersonic wake problem without first determining the detailed motion in the base flow region.

(vi) The theory confirms many of the qualitative predictions of the integral techniques. However, the present exact treatment of the boundary-layer equations indicates that the inaccuracy of the integral averaging procedures *per se* can be more important than detailed variations in the profile shapes.

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Appendix

The expressions for B_1 and B_2 referred to in (3.19) are

$$B_1 = \frac{Uh}{u^2}\left(\frac{-2(\mu u_y)_y}{u} + \frac{1}{P}\frac{(\mu h_y)_y}{h} + (\gamma-1)M_\infty^2\frac{\mu u_y^2}{h}\right)$$
$$+\frac{H}{u}\left(\frac{(\mu u_y)_y + h(\mu' u_y)_y}{u} - \frac{\mu_{yy}}{P} - (\gamma-1)M_\infty^2\mu' u_y^2\right)$$
$$+\frac{U_y}{u}\left(\frac{\mu_y h}{u} - 2(\gamma-1)M_\infty^2\mu u_y\right) + U_{yy}\frac{\mu h}{u^2}$$
$$+\frac{H_y}{u}\left(\frac{\mu' u_y h}{u} - \frac{2}{P}\mu_y\right) - \frac{H_{yy}}{P}\frac{\mu}{u}, \tag{A 1}$$

and

$$B_2 = \frac{\bar{U}h}{u^2}\left(\frac{-2(\mu u_y)_y}{u} + \frac{1}{P}\frac{(\mu h_y)_y}{h} + (\gamma-1)M_\infty^2\frac{\mu u_y^2}{h}\right)$$
$$+\frac{\bar{H}}{u}\left(\frac{(\mu u_y)_y + h(\mu' u_y)_y}{u} - \frac{\mu_{yy}}{Pr} - (\gamma-1)M_\infty^2\mu' u_y^2\right)$$
$$+\frac{\bar{U}_y}{u}\left(\frac{\mu_y h}{u} - 2(\gamma-1)M_\infty^2\mu u_y\right) + \bar{U}_{yy}\frac{\mu h}{u^2}$$
$$+\frac{\bar{H}_y}{u}\left(\frac{\mu' u_y h}{u} - \frac{2}{P}\mu_y\right) - \frac{\bar{H}_{yy}}{P}\frac{\mu}{u}, \tag{A 2}$$

where

$$\mu' = \frac{d\mu}{dh}.$$

The expressions for N_1, N_2, D_1 and D_2 referred to in (3.28) are

$$N_1(x^*) = \phi_e^* + (1+\phi_e^*)\frac{\sqrt{(M_e^{*2}-1)}}{\gamma M_e^{*2}} + \frac{1}{R_{\infty L}}\int_0^{\delta(x^*)} B_1\,dy, \tag{A 3}$$

$$N_2(x^*) = \frac{1}{R_{\infty L}}\int_0^{\delta(x^*)} B_2\,dy, \tag{A 4}$$

$$D_1(x^*) = \int_0^{\delta(x^*)} \frac{1}{\gamma M^2}\left(\frac{2U}{u} - \frac{H}{h}\right)dy + (\delta-\bar{\delta})\left(\frac{M_e^{*2}-1}{\gamma M_e^{*2}}\right)$$
$$\times\left(\frac{M_e^{*2} - 1 + \gamma(M_e^*\phi_e^*)^2 - (M_e^{*2}-1)^{\frac{1}{2}}(\phi_e^* + (\gamma-1)M_e^{*2})}{\gamma p_e^* M_e^{*2}(1+\phi_e^{*2})}\right), \tag{A 5}$$

and

$$D_2(x^*) = \int_0^{\delta(x^*)} \frac{1}{\gamma M^2}\left(\frac{2\bar{U}}{u} - \frac{\bar{H}}{h}\right)dy + \left(\frac{M_e^{*2}-1}{\gamma M_e^{*2}}\right)\phi_e^*, \tag{A 6}$$

where $\phi_e^* = v_e^*/u_e^* = \tan\theta_e^*$.

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